

Plasmon resonance with finite frequencies: a validation of the quasi-static approximation for diametrically small inclusions

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Abstract

We study resonance for the Helmholtz equation with a finite frequency in a plasmonic material of negative dielectric constant in two and three dimensions. We show that the quasi-static approximation is valid for diametrically small inclusions. In fact, we quantitatively prove that if the diameter of a inclusion is small compared to the loss parameter, then resonance occurs exactly at eigenvalues of the Neumann-Poincaré operator associated with the inclusion.

AMS subject classifications.

Key words. Neumann-Poincaré operator, eigenvalues, Helmholtz equation, finite frequency, plasmon resonance, quasi-static limit

1 Introduction

Resonance phenomena is often observed in nanoscale particles whose material has a negative dielectric permittivity with a large wavelength in comparison with particle dimensions both experimentally and numerically [18]. It is known that such a resonance only occurs at certain frequencies. Noble metals such as gold and silver show a negative permittivity [19], and are called plasmonic materials. Recently, there has been considerable interest in the plasmon resonance and its various applications including invisibility cloaking, biomedical imaging and medical therapy; see, e.g., [1, 2, 3, 7, 8, 9, 12, 14, 15, 16, 18] and references therein.

It is known (see, e.g., [7, 8]) that in the quasi-static limit the plasmon resonance occurs at the eigenvalues of the Neumann-Poincaré operator associated with the inclusion. To be more precise, let D be a bounded simply connected domain in \mathbb{R}^d ($d = 2, 3$) whose boundary ∂D is $\mathcal{C}^{1,\alpha}$ for some $0 < \alpha < 1$. Suppose that D is occupied with a plasmonic material which has a dielectric constant $\epsilon_c + i\delta$, where $\epsilon_c < 0$ and $\delta > 0$ is the dissipation, and that the matrix $\mathbb{R} \setminus \overline{\Omega}$ has a dielectric constant $\epsilon_m > 0$. Hence, the distribution of the dielectric constant is given by

$$\epsilon_D = \begin{cases} \epsilon_c + i\delta, & \text{in } D, \\ \epsilon_m, & \text{in } \mathbb{R} \setminus \overline{D}. \end{cases} \quad (1.1)$$

The dielectric equation in the quasi-static limit is given by

$$\nabla \cdot \epsilon_D \nabla u_\delta = f. \quad (1.2)$$

It is proved (e.g., [7]) that when the source f is given by the polarizable dipole $a \cdot \nabla \delta_z$, the resonance occurs exactly when $\lambda(\epsilon_c/\epsilon_m)$ is an eigenvalue of the the Neumann-Poincaré (NP) operator associated with D (see the next section for the definition and spectral properties of the NP operator),

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in other words, $\|\nabla u_\delta\|_{L^2(D)} \rightarrow \infty$ as $\delta \rightarrow \infty$. Here,

$$\lambda(t) := \frac{t+1}{2(t-1)}. \quad (1.3)$$

When $\lambda(\epsilon_c/\epsilon_m)$ is an eigenvalue of the the NP operator, ϵ_c/ϵ_m is called the plasmon eigenvalue [8].

In this paper, we consider plasmon resonance for the Helmholtz operator $\nabla \cdot \epsilon_D \nabla + \omega_0^2$, when D is a diametrically small inclusion such as a nano-scale particle. Here ω_0 represents the non-zero (but fixed) frequency, and the parameters ϵ_c and δ are determined by ω_0 . We show that if the diameter s of D is much smaller than the dissipation parameter δ , then the resonance occurs exactly when $\lambda(\epsilon_c/\epsilon_m)$ is an eigenvalue of the NP operator on D , like the quasi-static limit case. So the result of this paper can be regarded as a validation of the quasi-static approximation for small inclusions. It is worth mentioning that a different validation of quasi-static approximation is proved in [3] by showing that the small volume asymptotic expansion of the far field for the Maxwell system holds away from the eigenvalues of the NP operator.

To describe results of this paper in a quantitative manner, let $D = s\Omega$, and let after scaling

$$\epsilon_\Omega = \begin{cases} \epsilon_c + i\delta, & \text{in } \Omega, \\ \epsilon_m, & \text{in } \mathbb{R} \setminus \overline{\Omega}. \end{cases} \quad (1.4)$$

We then consider

$$\nabla \cdot \epsilon_\Omega \nabla u_\delta + s^2 \omega_0^2 u_\delta = a \cdot \nabla \delta_z \quad \text{in } \mathbb{R}^d \quad (1.5)$$

satisfying the Sommerfeld radiation condition

$$\left| \frac{\partial u_\delta}{\partial r} - i\omega \epsilon_m^{-1/2} u_\delta \right| \leq C r^{-(d+1)/2} \quad \text{as } r = |x| \rightarrow \infty, \quad (1.6)$$

where $a \in \mathbb{R}^d$ is a constant vector and δ_z is the Dirac mass at $z \in \mathbb{R}^d \setminus \overline{\Omega}$. We characterize the resonance by the blow-up of $\|\nabla u_\delta\|_{L^2(\Omega)}$:

$$\|\nabla u_\delta\|_{L^2(\Omega)} \rightarrow \infty \quad \text{as } \delta \rightarrow +0. \quad (1.7)$$

where u_δ is the solution to (1.5).

We show that if s is much smaller than δ , more precisely, if $s\delta^{-1} \ll 1$ in three dimensions, and $s^2 |\ln s| \delta^{-1} \ll 1$ in two dimensions, then (1.7) takes place if and only if $\lambda(\epsilon_c/\epsilon_m)$ is an eigenvalue of the NP operator on Ω . Moreover, if $\lambda(\epsilon_c/\epsilon_m)$ is an eigenvalue, we obtain a quantitative estimate

$$\|\nabla u_\delta\|_{L^2(\Omega)} \approx \delta^{-1} \quad \text{as } \delta \rightarrow +0 \quad (1.8)$$

for most z (the location of the dipole source). See Theorem 4.3 and Theorem 4.5 for precise statements. It is worth mentioning that the spectrums of the NP operators on D and on Ω are the same.

The rest of this paper is organized as follows. In section 2 we review spectral properties of the NP operator. Section 3 is to derive necessary asymptotic formula for the Helmholtz operator at low frequencies and estimates for the H^1 -norm of the solution. The main results in three and two dimensions are presented and proved in subsection 4.1 and subsection 4.2, respectively.

While writing this paper (after completing major work) we received the paper [6] from Habib Ammari. There an asymptotic formula for the solution similar to (4.11) is derived in three dimensions when there are multiple small inclusions, using the same method as in this paper (the spectral properties of the NP operator). Then the formula is used to study enhancement of scattering and absorption, and super-resonance by plasmonic particles. Here in this paper we use the asymptotic formula to show resonance quantified by (1.8).

2 Preliminaries

Let Ω be a bounded domain with the Lipschitz boundary in \mathbb{R}^d , $d = 2, 3$. Throughout this paper $H^s(\partial\Omega)$ denotes the L^2 -Sobolev space on $\partial\Omega$ whose norm is expressed as $\|\cdot\|_s$. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing of $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$. Let $H_0^{-1/2}(\partial\Omega)$ be the space of $\psi \in H^{-1/2}(\partial\Omega)$ satisfying $\langle \psi, 1 \rangle = 0$.

Let $\Gamma(x)$ be the fundamental solution to the Laplacian on \mathbb{R}^d , $d = 2, 3$:

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi} \ln|x|, & d = 2, \\ -\frac{1}{4\pi} |x|^{-1}, & d = 3. \end{cases}$$

The single layer potential of $\varphi \in H^{-1/2}(\partial\Omega)$ for the Laplacian is defined by

$$\mathcal{S}[\varphi](x) = \int_{\partial\Omega} \Gamma(x - y) \varphi(y) d\sigma(y), \quad x \in \mathbb{R}^d.$$

It is well known (see e.g. [4]) that the following jump formula holds:

$$\partial_\nu \mathcal{S}[\varphi] \big|_{\pm}(x) = (\pm 1/2I + \mathcal{K}^*) [\varphi](x), \quad x \in \partial\Omega, \quad (2.1)$$

where \mathcal{K}^* is the Neumann-Poincaré (NP) operator defined by

$$\mathcal{K}^*[\varphi](x) = \text{p.v.} \int_{\partial\Omega} \partial_{\nu_x} \Gamma(x - y) \varphi(y) d\sigma(y), \quad x \in \partial\Omega. \quad (2.2)$$

Here ∂_ν denotes the outward normal derivative, the subscripts \pm the limit (to $\partial\Omega$) from outside and inside of Ω , respectively, and p.v. the Cauchy principal value.

It is proved in [13] (see also [10]) that the NP operator \mathcal{K}^* can be symmetrized using Plemelj's symmetrization principle:

$$\mathcal{SK}^* = \mathcal{K}\mathcal{S}. \quad (2.3)$$

In fact, if we define a new inner product on $H_0^{-1/2}(\partial\Omega)$ by

$$\langle \varphi, \psi \rangle_{\mathcal{H}^*} := -\langle \varphi, \mathcal{S}[\psi] \rangle, \quad (2.4)$$

where the right hand side of (2.4) is well-defined since \mathcal{S} maps $H^{-1/2}(\partial\Omega)$ into $H^{1/2}(\partial\Omega)$, then \mathcal{K}^* is self-adjoint with respect to this inner product. Let \mathcal{H}_0^* be the space $H_0^{-1/2}(\partial\Omega)$ equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}^*}$ and $\|\cdot\|_{\mathcal{H}^*}$ be the induced norm. It is known (see [11]) that $\|\cdot\|_{\mathcal{H}^*}$ is equivalent to the norm $\|\cdot\|_{-1/2}$:

$$\|\varphi\|_{\mathcal{H}^*} \approx \|\varphi\|_{-1/2} \quad (2.5)$$

for $\varphi \in H_0^{-1/2}(\partial\Omega)$. Here and throughout this paper $A \lesssim B$ means $A \leq CB$ for some constant C independent of parameters involved; $A \approx B$ means that $A \lesssim B$ and $A \gtrsim B$.

There is a nontrivial $\varphi_0 \in H^{-1/2}(\partial\Omega)$ such that

$$\mathcal{K}^*[\varphi_0] = \frac{1}{2} \varphi_0, \quad (2.6)$$

We note that $\mathcal{S}[\varphi_0]$ is constant, say c_0 , in Ω . In three dimensions, $c_0 \neq 0$, and hence $\mathcal{S} : H^{-1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$ is invertible. However, there are domains Ω in two dimensions such that $c_0 = 0$ (see [20]), which means \mathcal{S} is not invertible in general. We introduce a variance of the single layer potential, denoted by $\tilde{\mathcal{S}}$, by $\tilde{\mathcal{S}} = \mathcal{S}$ if $c_0 \neq 0$, and if $c_0 = 0$, then

$$\tilde{\mathcal{S}}[\varphi] = \begin{cases} \mathcal{S}[\varphi], & \text{if } \langle \varphi, 1 \rangle = 0, \\ 1, & \text{if } \varphi = \varphi_0. \end{cases}$$

Then $\tilde{\mathcal{S}}$ is a bijection from $H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$. Moreover, we have an extension of (2.3):

$$\tilde{\mathcal{S}}\mathcal{K}^* = \mathcal{K}\tilde{\mathcal{S}} \quad (2.7)$$

which enables us to extend the inner product (2.4) to $H^{-1/2}(\partial\Omega)$:

$$\langle \varphi, \psi \rangle_{\mathcal{H}^*} := -\langle \varphi, \tilde{\mathcal{S}}_{\partial\Omega}[\psi] \rangle. \quad (2.8)$$

We denote by \mathcal{H}^* the space $H^{-1/2}(\partial\Omega)$ equipped with the inner product (2.8). Then the symmetrization principle (2.7) makes \mathcal{K}^* self-adjoint on \mathcal{H}^* . We emphasize that the norm equivalence (2.5) is valid for $\varphi \in H^{-1/2}(\partial\Omega)$.

The spectrum $\sigma(\mathcal{K}^*)$ of the NP operator lies in $(-1/2, 1/2]$. Moreover, $\mathcal{K}_{\partial\Omega}^*$ is a compact operator on \mathcal{H}^* , when $\partial\Omega$ is $C^{1,\alpha}$ for some $\alpha > 0$. Therefore, we have the spectral decomposition of \mathcal{K}^* on \mathcal{H}^* :

$$\mathcal{K}^* = \frac{1}{2}\varphi_0 \otimes \varphi_0 + \sum_{n=1}^{\infty} \lambda_n \varphi_n \otimes \varphi_n = \sum_{n=0}^{\infty} \lambda_n \varphi_n \otimes \varphi_n, \quad (2.9)$$

where $\varphi_n \in \mathcal{H}^*$ is an eigenvector of \mathcal{K}^* corresponding to the eigenvalue $\lambda_n \in \mathbb{R}$ (counting multiplicities), with $1/2 = \lambda_0 > |\lambda_1| \geq |\lambda_2| \cdots \geq |\lambda_n| \geq \cdots \rightarrow 0$ as $n \rightarrow \infty$. We note that $\{\varphi_n\}_{n=0}^{\infty}$ is chosen to be an orthonormal basis on \mathcal{H}^* (φ_0 is normalized so that $\|\varphi_0\|_{\mathcal{H}^*} = 1$).

Define an inner product

$$\langle f, g \rangle_{\mathcal{H}} := -\langle f, \tilde{\mathcal{S}}^{-1}[g] \rangle \quad (2.10)$$

on $H^{1/2}(\partial\Omega)$, and denote by \mathcal{H} the space $H^{1/2}(\partial\Omega)$ equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Then $\tilde{\mathcal{S}}$ is a unitary operator from \mathcal{H}^* to \mathcal{H} , and hence $\{\tilde{\mathcal{S}}[\varphi_n], n = 0, 1, \dots\}$ is an orthonormal basis of \mathcal{H} . Let \mathcal{H}_0 be the subspace of \mathcal{H} spanned by $\{\tilde{\mathcal{S}}[\varphi_n], n = 1, \dots\}$. Then $\mathcal{S} : \mathcal{H}_0^* \rightarrow \mathcal{H}_0$ is a bijection. We emphasize that the norm $\|\cdot\|_{\mathcal{H}}$ is equivalent to $\|\cdot\|_{1/2}$.

For $\varphi \in \mathcal{H}^*$, we write

$$\hat{\varphi}(n) := \langle \varphi, \varphi_n \rangle_{\mathcal{H}^*}, \quad n = 0, 1, 2, \dots, \quad (2.11)$$

so that

$$\varphi = \sum_{n=0}^{\infty} \hat{\varphi}(n) \varphi_n (= \hat{\varphi}(0) \varphi_0 + \varphi'), \quad \|\varphi\|_{\mathcal{H}^*}^2 = |\hat{\varphi}(0)|^2 + \|\varphi'\|_{\mathcal{H}^*}^2. \quad (2.12)$$

For $f \in \mathcal{H}$, we define

$$\check{f}(n) := \langle f, \tilde{\mathcal{S}}[\varphi_n] \rangle_{\mathcal{H}}, \quad n = 0, 1, 2, \dots, \quad (2.13)$$

so that

$$f = \sum_{n=0}^{\infty} \check{f}(n) \tilde{\mathcal{S}}[\varphi_n] (= \check{f}(0) \tilde{\mathcal{S}}[\varphi_0] + f'), \quad \|f\|_{\mathcal{H}}^2 = |\check{f}(0)|^2 + \|f'\|_{\mathcal{H}}^2. \quad (2.14)$$

We refer to [7] and references therein for more details on the preliminaries presented in this section.

Finally, we denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from a Banach space X to a Banach space Y ; in particular, $\mathcal{L}(X)$ is the space of bounded linear operators on a Banach space X .

3 Asymptotic expansion at low frequencies

Let $\omega = s\omega_0$ from now on to make notation short. A fundamental solution $\Gamma^{\omega}(x)$ to the Helmholtz operator $\Delta + \omega^2$ in \mathbb{R}^d is a solution of

$$(\Delta + \omega^2)\Gamma^{\omega} = \delta_0, \quad (3.1)$$

where δ_0 is the Dirac function at 0. Among solutions to (3.1), we seek a solution satisfying the Sommerfeld radiation condition

$$\left| \frac{\partial \Gamma^{\omega}}{\partial r} - i\omega \Gamma^{\omega} \right| \leq C r^{-(d+1)/2} \quad \text{as } r = |x| \rightarrow \infty. \quad (3.2)$$

Then, it is given by

$$\Gamma^\omega(x) = \begin{cases} -\frac{i}{4}H_0^1(\omega|x|) & \text{if } d = 2, \\ -\frac{1}{4\pi}\frac{e^{i\omega|x|}}{|x|} & \text{if } d = 3, \end{cases} \quad (3.3)$$

where $H_0^1(z)$ is the Hankel function of the first kind of order 0.

For the subsequent use, we consider the asymptotic expansion of the fundamental solution $\Gamma^\omega(x)$ as $\omega \rightarrow +0$. When $d = 2$, we recall the behavior of the Hankel function $H_0^1(z)$ near $z = 0$ (see, e.g., [17]):

$$-\frac{i}{4}H_0^1(\omega|x|) = \frac{1}{2\pi} \ln|x| + \tau + \sum_{n=1}^{\infty} (b_n \ln(\omega|x|) + c_n) (\omega|x|)^{2n}, \quad (3.4)$$

where

$$b_n = \frac{(-1)^n}{2\pi} \frac{1}{2^{2n} (n!)^2}, \quad c_n = -b_n \left(\gamma - \ln 2 - \frac{\pi i}{2} - \sum_{j=1}^n \frac{1}{j} \right)$$

and

$$\tau = \frac{1}{2\pi} (\ln \omega + \gamma - \ln 2) - \frac{i}{4} \quad (3.5)$$

(γ is the Euler constant). So we have

$$\Gamma^\omega(x) = \Gamma(x) + \tau + \omega^2 \ln \omega K_2^\omega(x) \quad (3.6)$$

as $\omega \rightarrow +0$ (see also [5]). The definition of $K_2^\omega(x)$ is obvious. When $d = 3$, one can easily see that

$$-\frac{1}{4\pi} \frac{e^{i\omega|x|}}{|x|} = -\frac{1}{4\pi} \frac{1}{|x|} - \frac{i\omega}{4\pi} \sum_{n=1}^{\infty} \frac{(i\omega|x|)^{n-1}}{n!}, \quad (3.7)$$

which implies that

$$\Gamma^\omega(x) = \Gamma(x) + \omega K_3^\omega(x). \quad (3.8)$$

Let us observe a regularity property of the function $K_d^\omega(x)$ ($d = 2, 3$) for later purpose. Let ω_1 be a small positive number. Then there is a constant C independent of $\omega \leq \omega_1$ such that

$$\int_{\Omega} \int_{\partial\Omega} |\partial_x^\alpha K_d^\omega(x-y)|^2 d\sigma(y) dx \leq C \quad (3.9)$$

for all $\alpha = (\alpha_1, \dots, \alpha_d)$ such that $|\alpha| \leq 2$. Here ∂_x^α is the partial derivative with respect to x . Moreover, $\nabla K_3^\omega(x)$ gains ω and it holds that

$$\frac{1}{\omega} \int_{\Omega} \int_{\partial\Omega} |\partial_x^\alpha \nabla_x K_3^\omega(x-y)|^2 d\sigma(y) dx \leq C \quad (3.10)$$

for all $|\alpha| \leq 1$.

The single layer potential of $\varphi \in H^{-1/2}(\partial\Omega)$ for the Helmholtz operator $\Delta + \omega^2$ is defined by

$$\mathcal{S}^\omega[\varphi](x) = \int_{\partial\Omega} \Gamma^\omega(x-y) \varphi(y) d\sigma(y), \quad x \in \mathbb{R}^d. \quad (3.11)$$

We note that $\mathcal{S}^\omega[\varphi](x)$ satisfies the Sommerfeld radiation condition (3.2) (see [5]). Let \mathcal{R}_d^ω ($d = 2, 3$) be the integral operator defined by K_d^ω , namely,

$$\mathcal{R}_d^\omega[\varphi](x) = \int_{\partial\Omega} K_d^\omega(x-y) \varphi(y) d\sigma(y), \quad x \in \mathbb{R}^d. \quad (3.12)$$

Then, we obtain from (3.6) and (3.8) that

$$\mathcal{S}^\omega = \begin{cases} \mathcal{S} + \tau \langle \cdot, 1 \rangle + \omega^2 \ln \omega \mathcal{R}_2^\omega & \text{if } d = 2, \\ \mathcal{S} + \omega \mathcal{R}_3^\omega & \text{if } d = 3. \end{cases} \quad (3.13)$$

Analogously to (2.1), the following jump formula holds:

$$\partial_\nu \mathcal{S}^\omega[\varphi]|_{\pm}(x) = (\pm 1/2I + (\mathcal{K}^\omega)^*)[\varphi](x), \quad x \in \partial\Omega, \quad (3.14)$$

where $(\mathcal{K}^\omega)^*$ is defined by

$$(\mathcal{K}^\omega)^*[\varphi](x) = \int_{\partial\Omega} \partial_{\nu_x} \Gamma^\omega(x-y) \varphi(y) d\sigma(y), \quad x \in \partial\Omega.$$

For $d = 2, 3$, let

$$\mathcal{Q}_d^\omega[\varphi](x) := \begin{cases} \partial_\nu \mathcal{R}_2^\omega[\varphi](x), & d = 2, \\ \frac{1}{\omega} \partial_\nu \mathcal{R}_3^\omega[\varphi](x), & d = 3, \end{cases} \quad x \in \partial\Omega. \quad (3.15)$$

Then, we have

$$(\mathcal{K}^\omega)^* = \begin{cases} \mathcal{K}^* + \omega^2 \ln \omega \mathcal{Q}_2^\omega & \text{if } d = 2, \\ \mathcal{K}^* + \omega^2 \mathcal{Q}_3^\omega & \text{if } d = 3. \end{cases} \quad (3.16)$$

We now investigate the mapping property of \mathcal{R}_d^ω and \mathcal{Q}_d^ω . By Cauchy-Schwartz inequality we see from (3.9) that

$$\|\mathcal{R}_d^\omega[\varphi]\|_{W^{2,2}(\Omega)} \leq C \|\varphi\|_{L^2(\partial\Omega)}.$$

We also see from (3.10) that

$$\frac{1}{\omega} \|\nabla \mathcal{R}_3^\omega[\varphi]\|_{W^{1,2}(\Omega)} \leq C \|\varphi\|_{L^2(\partial\Omega)}.$$

By trace theorem, \mathcal{R}_d^ω maps $L^2(\partial\Omega)$ into $H^{3/2}(\partial\Omega)$, and \mathcal{Q}_d^ω maps $L^2(\partial\Omega)$ into $H^{1/2}(\partial\Omega)$. By duality, \mathcal{R}_d^ω maps $H^{-3/2}(\partial\Omega)$ into $L^2(\partial\Omega)$, and $H^{-1/2}(\partial\Omega)$ into $H^1(\partial\Omega)$ by interpolation. Likewise we see that \mathcal{Q}_d^ω maps $H^{-1/2}(\partial\Omega)$ into $L^2(\partial\Omega)$. We summarize these properties in the following lemma.

Lemma 3.1. *For a given small positive number ω_1 , there exists a constant C independent of $\omega \leq \omega_1$ such that*

$$\|\mathcal{R}_d^\omega[\varphi]\|_1 \leq C \|\varphi\|_{-1/2} \quad (3.17)$$

and

$$\|\mathcal{Q}_d^\omega[\varphi]\|_0 \leq C \|\varphi\|_{-1/2} \quad (3.18)$$

for all $\varphi \in H^{-1/2}(\partial\Omega)$.

Proposition 3.2. *Let $\varphi \in \mathcal{H}^*$ and $\varphi = \varphi' + \hat{\varphi}(0)\varphi_0$ be its orthogonal decomposition where $\varphi' \in \mathcal{H}_0^*$. The following estimates hold:*

(i) *If $d = 2$, then*

$$\|\varphi'\|_{\mathcal{H}^*}^2 - |\omega \ln \omega|^2 |\hat{\varphi}(0)|^2 \lesssim \|\nabla \mathcal{S}_{\partial\Omega}^\omega[\varphi]\|_{L^2(\Omega)}^2 \lesssim \|\varphi'\|_{\mathcal{H}^*}^2 + |\omega \ln \omega|^2 |\hat{\varphi}(0)|^2. \quad (3.19)$$

(ii) *If $d = 3$, then*

$$\|\varphi'\|_{\mathcal{H}^*}^2 - |\omega \ln \omega|^2 |\hat{\varphi}(0)|^2 \lesssim \|\nabla \mathcal{S}_{\partial\Omega}^\omega[\varphi]\|_{L^2(\Omega)}^2 \lesssim \|\varphi'\|_{\mathcal{H}^*}^2 + |\omega \ln \omega|^2 |\hat{\varphi}(0)|^2. \quad (3.20)$$

Proof. We only prove (3.19) since three dimensional case can be proved in a similar way.

We have from Gauss's divergence theorem

$$\int_{\Omega} |\nabla \mathcal{S}^{\omega}[\varphi]|^2 dx + \int_{\Omega} \mathcal{S}^{\omega}[\varphi] \overline{\Delta \mathcal{S}^{\omega}[\varphi]} dx = \int_{\partial\Omega} \mathcal{S}^{\omega}[\varphi] \overline{\partial_{\nu} \mathcal{S}^{\omega}[\varphi]}_{-} d\sigma. \quad (3.21)$$

Since $\Delta \mathcal{S}^{\omega}[\varphi] = -\omega^2 \mathcal{S}^{\omega}[\varphi]$, we have

$$\int_{\Omega} |\nabla \mathcal{S}^{\omega}[\varphi]|^2 dx = \omega^2 \int_{\Omega} |\mathcal{S}^{\omega}[\varphi]|^2 dx + \int_{\partial\Omega} \mathcal{S}^{\omega}[\varphi] \overline{\partial_{\nu} \mathcal{S}^{\omega}[\varphi]}_{-} d\sigma. \quad (3.22)$$

One can see from (3.13) and Lemma 3.1 that

$$\int_{\Omega} |\mathcal{S}^{\omega}[\varphi]|^2 dx \lesssim |\ln \omega| \|\varphi\|_{-1/2} \lesssim |\ln \omega| \|\varphi\|_{\mathcal{H}^*}, \quad (3.23)$$

since $|\tau| \lesssim |\ln \omega|$. The last inequality holds because of (2.5).

Using the jump formula (3.14) we have

$$\int_{\partial\Omega} \mathcal{S}^{\omega}[\varphi] \overline{\partial_{\nu} \mathcal{S}^{\omega}[\varphi]}_{-} d\sigma = \int_{\partial\Omega} \mathcal{S}^{\omega}[\varphi] \overline{(-1/2I + (\mathcal{K}^{\omega})^*)[\varphi]} d\sigma.$$

One then see from (3.13) and (3.16) that

$$\begin{aligned} & \int_{\partial\Omega} \mathcal{S}^{\omega}[\varphi] \overline{\partial_{\nu} \mathcal{S}^{\omega}[\varphi]}_{-} d\sigma \\ &= \int_{\partial\Omega} \mathcal{S}[\varphi] \overline{(-1/2I + \mathcal{K}^*)[\varphi]} d\sigma + \tau \langle \varphi, 1 \rangle \int_{\partial\Omega} \overline{(-1/2I + \mathcal{K}^*)[\varphi]} d\sigma + \omega^2 \ln \omega E \end{aligned} \quad (3.24)$$

where

$$E = \int_{\partial\Omega} \mathcal{R}_2^{\omega}[\varphi] \overline{(-1/2I + (\mathcal{K}^{\omega})^*)[\varphi]} d\sigma + \int_{\partial\Omega} \mathcal{S}^{\omega}[\varphi] \overline{\mathcal{Q}_2^{\omega}[\varphi]} d\sigma + \tau \langle \varphi, 1 \rangle \int_{\partial\Omega} \overline{\mathcal{Q}_2^{\omega}[\varphi]} d\sigma.$$

Using (3.13) and Lemma 3.1 one can show that

$$|E| \leq C |\ln \omega| \|\varphi\|_{\mathcal{H}^*}^2 \quad (3.25)$$

for some constant C independent of $\omega \leq \omega_1$. In fact, we have from (3.5)

$$\begin{aligned} |E| &\leq \|\mathcal{R}_2^{\omega}[\varphi]\|_{1/2} \|(-1/2I + (\mathcal{K}^{\omega})^*)[\varphi]\|_{-1/2} + \|\mathcal{S}^{\omega}[\varphi]\|_{1/2} \|\mathcal{Q}_2^{\omega}[\varphi]\|_{-1/2} + \tau \|\varphi\|_{-1/2} \|\mathcal{Q}_2^{\omega}[\varphi]\|_0 \\ &\leq C |\ln \omega| \|\varphi\|_{-1/2}^2. \end{aligned}$$

Since $\mathcal{K}[1] = 1/2$, we have

$$\int_{\partial\Omega} \overline{(-1/2I + \mathcal{K}^*)[\varphi]} d\sigma = \int_{\partial\Omega} (-1/2I + \mathcal{K})[1] \overline{\varphi} d\sigma = 0. \quad (3.26)$$

On the other hand, since $\mathcal{K}^*[\varphi_0] = 1/2\varphi_0$, we have

$$\int_{\partial\Omega} \mathcal{S}[\varphi] \overline{(-1/2I + \mathcal{K}^*)[\varphi]} d\sigma = \int_{\partial\Omega} \mathcal{S}[\varphi'] \overline{(-1/2I + \mathcal{K}^*)[\varphi']} d\sigma.$$

Using $\varphi' = \sum_{n=1}^{\infty} \hat{\varphi}(n) \varphi_n$, we have

$$\int_{\partial\Omega} \mathcal{S}[\varphi] \overline{(-1/2I + \mathcal{K}^*)[\varphi]} d\sigma = \sum_{n,m=1}^{\infty} (-1/2 + \lambda_l) \hat{\varphi}(n) \overline{\hat{\varphi}(m)} \int_{\partial\Omega} \mathcal{S}[\varphi_j] \overline{\varphi_l} d\sigma.$$

Since $\int_{\partial\Omega} \mathcal{S}[\varphi_n] \overline{\varphi_m} d\sigma = -\langle \varphi_n, \varphi_m \rangle_{\mathcal{H}^*} = -\delta_{nm}$ (the Kronecker's delta), we have

$$\int_{\partial\Omega} \mathcal{S}[\varphi] \overline{(-1/2I + \mathcal{K}^*)[\varphi]} d\sigma = \sum_{n=1}^{\infty} (\lambda_n - 1/2) |\hat{\varphi}(n)|^2.$$

So we have

$$\left| \int_{\partial\Omega} \mathcal{S}[\varphi] \overline{(-1/2I + \mathcal{K}^*)[\varphi]} d\sigma \right| \approx \|\varphi'\|_{\mathcal{H}^*}^2. \quad (3.27)$$

Combining (3.24)-(3.27) we obtain

$$\|\varphi'\|_{\mathcal{H}^*}^2 - |\omega \ln \omega|^2 \|\varphi\|_{\mathcal{H}^*}^2 \lesssim \left| \int_{\partial\Omega} \mathcal{S}^\omega[\varphi] \overline{\partial_\nu \mathcal{S}^\omega[\varphi]} d\sigma \right| \lesssim \|\varphi'\|_{\mathcal{H}^*}^2 + |\omega \ln \omega|^2 \|\varphi\|_{\mathcal{H}^*}^2,$$

which together with (3.21) and (3.23) yields (3.19). \square

4 Analysis of resonance

From now on, we assume that $\epsilon_m = 1$ without loss of generality.

Set $k_m = \omega (= s\omega_0)$ and

$$k_c^2 = \frac{\omega^2}{\epsilon_c + i\delta}, \quad \Re k_c > 0, \quad \Im k_c < 0.$$

Since

$$k_c = \omega (\epsilon_c + i\delta)^{-1/2} \simeq -i \frac{\omega}{\sqrt{|\epsilon_c|}} \left(1 - i \frac{\delta}{2\epsilon_c} \right),$$

we assume for simplicity

$$k_c = -i \frac{\omega}{\sqrt{|\epsilon_c|}} \left(1 - i \frac{\delta}{2\epsilon_c} \right). \quad (4.1)$$

Then the problem (1.5) can be written as

$$\begin{cases} \Delta u_\delta + k_c^2 u_\delta = 0 & \text{in } \Omega, \\ \Delta u_\delta + \omega^2 u_\delta = a \cdot \nabla \delta_z & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ u_\delta|_- - u_\delta|_+ = 0 & \text{on } \partial\Omega, \\ (\epsilon_c + i\delta) \partial_\nu u_\delta|_- - \partial_\nu u_\delta|_+ = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.2)$$

under the Sommerfeld radiation condition (1.6).

Let

$$F_z(x) := -a \cdot \nabla_x \Gamma^\omega(x - z). \quad (4.3)$$

Then, the solution u_δ can be represented as

$$u_\delta(x) = \begin{cases} \mathcal{S}^{k_c}[\varphi_\delta](x), & x \in \Omega, \\ F_z(x) + \mathcal{S}^\omega[\psi_\delta](x), & x \in \mathbb{R}^d \setminus \Omega \end{cases} \quad (4.4)$$

for some $\varphi_\delta, \psi_\delta \in \mathcal{H}^*$. In view of transmission conditions on $\partial\Omega$ (the third and fourth conditions in (4.2)), $(\varphi_\delta, \psi_\delta)$ should solve the following system of integral equations:

$$\begin{cases} \mathcal{S}^{k_c}[\varphi_\delta] - \mathcal{S}^\omega[\psi_\delta] = F_z, \\ (\epsilon_c + i\delta) \partial_\nu \mathcal{S}^{k_c}[\varphi_\delta]|_- - \partial_\nu \mathcal{S}^\omega[\psi_\delta]|_+ = \partial_\nu F_z, \end{cases} \quad \text{on } \partial\Omega. \quad (4.5)$$

Let $X := \mathcal{H}^* \times \mathcal{H}^*$ and $Y := \mathcal{H} \times \mathcal{H}^*$, and define an operator $A_\delta^s : X \rightarrow Y$ by

$$A_\delta^s = \begin{bmatrix} \mathcal{S}^{k_c} & -\mathcal{S}^\omega \\ (\epsilon_c + i\delta) \partial_\nu \mathcal{S}^{k_c}|_- & -\partial_\nu \mathcal{S}^\omega|_+ \end{bmatrix}. \quad (4.6)$$

Then we can rewrite (4.5) as

$$A_\delta^s \begin{bmatrix} \varphi_\delta \\ \psi_\delta \end{bmatrix} = \begin{bmatrix} F_z \\ \partial_\nu F_z \end{bmatrix}. \quad (4.7)$$

The solvability of (4.5) is equivalent to the invertibility of A_δ^s . We will investigate the behavior of the norm $(A_\delta^s)^{-1}$ as $\delta \rightarrow +0$.

4.1 Three dimensions

We deal with the three dimensional case first since it is easier.

We split A_δ^s into two parts: $A_\delta^s = A_\delta + T_\delta^s$, where

$$A_\delta = \begin{bmatrix} \mathcal{S} & -\mathcal{S} \\ (\epsilon_c + i\delta)(-1/2I + \mathcal{K}^*) & -(1/2I + \mathcal{K}^*) \end{bmatrix}. \quad (4.8)$$

Then we can infer from (3.13), (3.16) and Lemma 3.1 that

$$\|T_\delta^s\|_{\mathcal{L}(X, Y)} \lesssim \omega. \quad (4.9)$$

Lemma 4.1. *For $f \in \mathcal{H}$ and $g \in \mathcal{H}^*$, the solution to*

$$A_\delta \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \quad (4.10)$$

is given by

$$\varphi = \sum_{n=0}^{\infty} \frac{\hat{g}(n) - (1/2 + \lambda_n)\check{f}(n)}{(\epsilon_c - 1)(\lambda_n - \lambda(\epsilon_c)) + i\delta(\lambda_n - \frac{1}{2})} \varphi_n \quad (4.11)$$

and

$$\psi = \varphi - \mathcal{S}^{-1}[f]. \quad (4.12)$$

Proof. The equation (4.10) can be written as

$$\begin{cases} \mathcal{S}[\varphi] - \mathcal{S}[\psi] = f, \\ (\epsilon_c + i\delta)(-1/2I + \mathcal{K}^*)[\varphi] - (1/2I + \mathcal{K}^*)[\psi] = g, \end{cases} \quad \text{on } \partial\Omega.$$

Since $\mathcal{S} : \mathcal{H}^* \rightarrow \mathcal{H}$ is invertible in three dimensions, we have

$$\psi = \varphi - \mathcal{S}^{-1}[f]. \quad (4.13)$$

Substituting this into the second equation, we obtain

$$(-1/2(\epsilon_c + i\delta + 1)I + (\epsilon_c + i\delta - 1)\mathcal{K}^*)[\varphi] = g - (1/2I + \mathcal{K}^*)\mathcal{S}^{-1}[f].$$

We then use the spectral decomposition (2.12) to obtain

$$\varphi = \sum_{n=0}^{\infty} \frac{a_n}{-1/2(\epsilon_c + i\delta + 1) + (\epsilon_c + i\delta - 1)\lambda_n} \varphi_n$$

where

$$a_n = \hat{g}(n) - \langle (1/2I + \mathcal{K}^*)\mathcal{S}^{-1}[f], \varphi_n \rangle_{\mathcal{H}^*}.$$

Since $f = \sum_{j=0}^{\infty} \check{f}(j)\mathcal{S}[\varphi_j]$, we have

$$\langle (1/2I + \mathcal{K}^*)\mathcal{S}^{-1}[f], \varphi_n \rangle_{\mathcal{H}^*} = \sum_{j=0}^{\infty} \check{f}(j) \langle (1/2I + \mathcal{K}^*)[\varphi_j], \varphi_n \rangle_{\mathcal{H}^*} = (1/2 + \lambda_n)\check{f}(n).$$

This completes the proof. \square

As a consequence of Theorem 4.1 we obtain the following corollary.

Corollary 4.2. *Suppose that $\epsilon_c \neq -1$, and let (φ, ψ) be the solution of (4.10). Then the following hold for sufficiently small δ :*

$$(i) \quad \|(A_\delta^0)^{-1}\|_{\mathcal{L}(Y, X)} \lesssim \delta^{-1}.$$

(ii) *If $\lambda(\epsilon_c) \neq \lambda_n$ for any n , then $\|(A_\delta^0)^{-1}\|_{\mathcal{L}(Y, X)} \leq C$ for some C depending on ϵ_c .*

(iii) *If $\lambda(\epsilon_c) = \lambda_n$ for some $n \neq 0$, then $\|\varphi'\|_{\mathcal{H}^*} \gtrsim |a_n| \delta^{-1}$, where $a_n = \hat{g}(n) - (1/2 + \lambda_n) \check{f}(n)$.*

Proof. Since

$$\frac{1}{|(\epsilon_c - 1)(\lambda_n - \lambda(\epsilon_c)) + i\delta(\lambda_n - \frac{1}{2})|} \lesssim \delta^{-1},$$

we have from (4.11) that

$$\|\varphi\|_{\mathcal{H}^*}^2 \lesssim \delta^{-2} \sum_{n=0}^{\infty} |\hat{g}(n) - (1/2 + \lambda_n) \check{f}(n)|^2 \lesssim \delta^{-2} (\|f\|_{\mathcal{H}}^2 + \|g\|_{\mathcal{H}^*}^2).$$

We have from (4.12) that

$$\|\psi\|_{\mathcal{H}^*}^2 \lesssim \delta^{-2} (\|f\|_{\mathcal{H}}^2 + \|g\|_{\mathcal{H}^*}^2).$$

This proves (i).

Since $\epsilon_c \neq -1$, $\lambda(\epsilon_c) \neq 0$. If $\lambda(\epsilon_c) \neq \lambda_n$ for any n , then $|\lambda_n - \lambda(\epsilon_c)| \geq C$ for some $C > 0$. So we have

$$\frac{1}{|(\epsilon_c - 1)(\lambda_n - \lambda(\epsilon_c)) + i\delta(\lambda_n - \frac{1}{2})|} \lesssim 1,$$

and hence

$$\|\varphi\|_{\mathcal{H}^*}^2 \lesssim \sum_{n=0}^{\infty} |\hat{g}(n) - (1/2 + \lambda_n) \check{f}(n)|^2 \lesssim \|f\|_{\mathcal{H}}^2 + \|g\|_{\mathcal{H}^*}^2$$

and

$$\|\psi\|_{\mathcal{H}^*}^2 \lesssim \|f\|_{\mathcal{H}}^2 + \|g\|_{\mathcal{H}^*}^2.$$

This proves (ii).

If $\lambda(\epsilon_c) = \lambda_n$ for some $n \neq 0$, then we have

$$\frac{1}{|(\epsilon_c - 1)(\lambda_n - \lambda(\epsilon_c)) + i\delta(\lambda_n - \frac{1}{2})|} \gtrsim \delta^{-1}$$

Therefore we have

$$\|\varphi'\|_{\mathcal{H}^*} \geq |\hat{\varphi}(n)| \gtrsim \delta^{-1} |a_n|.$$

This completes the proof. \square

The following is the main theorem of this paper in three dimensions.

Theorem 4.3. *Suppose $d = 3$ and assume*

$$s\delta^{-1} \leq c \tag{4.14}$$

for sufficiently small c . Let u_δ be the solution to (1.5).

(i) *If $\lambda(\epsilon_c/\epsilon_m) \neq \lambda_n$ for any n , then there is C independent of δ (may depend on ϵ_c/ϵ_m) such that*

$$\|\nabla u_\delta\|_{L^2(\Omega)} \leq C. \tag{4.15}$$

(iii) *If $\lambda(\epsilon_c/\epsilon_m) = \lambda_n$ for some $n \neq 0$, let z be such that $a \cdot \nabla \mathcal{S}[\varphi_n](z) \neq 0$. Then*

$$\|\nabla u_\delta\|_{L^2(\Omega)} \approx \delta^{-1} \tag{4.16}$$

as $\delta \rightarrow +0$.

Proof. We still assume $\epsilon_m = 1$. Since $A_\delta^s = A_\delta + T_\delta^s = A_\delta(I + (A_\delta)^{-1}T_\delta^s)$, it follows from (4.7) that

$$\Phi_\delta = (I + (A_\delta)^{-1}T_\delta^s)^{-1}(A_\delta)^{-1}[\mathbf{F}],$$

where

$$\Phi_\delta = \begin{bmatrix} \varphi_\delta \\ \psi_\delta \end{bmatrix} \quad \text{and} \quad \mathbf{F} = \begin{bmatrix} F_z \\ \partial_\nu F_z \end{bmatrix}.$$

We see from (4.9) and Corollary 4.2 (i) that

$$\|(A_\delta)^{-1}T_\delta^s\|_{\mathcal{L}(X)} \lesssim \delta^{-1}s.$$

So, if $s\delta^{-1}$ is sufficiently small, then we have

$$\|\Phi_\delta - (A_\delta)^{-1}[\mathbf{F}]\|_X \lesssim \delta^{-1}s\|(A_\delta)^{-1}[\mathbf{F}]\|_X. \quad (4.17)$$

If $\lambda(\epsilon_c) \neq \lambda_n$ for any n , then we infer from Corollary 4.2 (ii) that

$$\|\Phi_\delta\|_X \leq C\|\mathbf{F}\|_Y.$$

So, we have from (4.4) and (3.20)

$$\|\nabla u_\delta\|_{L^2(\Omega)} = \|\nabla \mathcal{S}^{k_c}[\varphi_\delta]\|_{L^2(\Omega)} \lesssim \|\varphi_\delta\|_{\mathcal{H}^*} \leq C$$

regardless of δ .

Suppose that $\lambda(\epsilon_c) = \lambda_n$ for some $n \neq 0$. Let $(A_\delta)^{-1}[\mathbf{F}] = (\varphi_1, \psi_1)^T$. Then Corollary 4.2 (iii) shows that

$$\|\varphi'_1\|_{\mathcal{H}^*} \gtrsim |a_n|\delta^{-1},$$

where

$$a_n = (\widehat{\partial_\nu F_z})(n) - (1/2 + \lambda_n)\check{F}_z(n). \quad (4.18)$$

It then follows from (4.17) that

$$\|\varphi'_\delta\|_{\mathcal{H}^*} \gtrsim \|\varphi'_1\|_{\mathcal{H}^*} - \delta^{-1}s\|(A_\delta)^{-1}[\mathbf{F}]\|_X \gtrsim |a_n|\delta^{-1}$$

if $a_n \neq 0$ for sufficiently small δ . Thus we obtain from (3.20) that

$$\|\nabla u_\delta\|_{L^2(\Omega)} = \|\nabla \mathcal{S}^{k_c}[\varphi_\delta]\|_{L^2(\Omega)} \gtrsim |a_n|\delta^{-1} - s|\hat{\varphi}_\delta(0)|. \quad (4.19)$$

We now show that $|\hat{\varphi}_\delta(0)|$ is bounded, and $a_n \neq 0$ for generic z 's. For that purpose we write A_δ^ω as $A_\delta^\omega = (I + T_\delta^s(A_\delta^0)^{-1})A_\delta^0$ so that (4.7) takes the form

$$A_\delta^0[\Phi_\delta] = (I + T_\delta^s(A_\delta^0)^{-1})^{-1}[\mathbf{F}] \quad (4.20)$$

Let $(I + T_\delta^s(A_\delta^0)^{-1})^{-1}[\mathbf{F}] = (f, g)^T$. Then since $\|T_\delta^s(A_\delta^0)^{-1}\|_{\mathcal{L}(Y)} \lesssim \delta^{-1}s$, we have $\|f\|_{\mathcal{H}} + \|g\|_{\mathcal{H}^*}$ is bounded. Since $\lambda_0 = 1/2$, we have according to (4.11)

$$|\hat{\varphi}_\delta(0)| = \left| \frac{\hat{g}(0) - \check{f}(0)}{(\epsilon_c - 1)(\frac{1}{2} - \lambda(\epsilon_c))} \right| \leq C.$$

Recall that $F_z(x) := -a \cdot \nabla_x \Gamma^\omega(x - z)$. According to (4.18) we have

$$\begin{aligned} a_n &= \langle \partial_\nu F_z, \varphi_n \rangle_{\mathcal{H}^*} - (1/2 + \lambda_n) \langle F_z, \mathcal{S}[\varphi_n] \rangle_{\mathcal{H}} \\ &= -\langle \partial_\nu F_z, \mathcal{S}[\varphi_n] \rangle + (1/2 + \lambda_n) \langle F_z, \varphi_n \rangle \\ &= \omega^2 \int_{\Omega} F_z \mathcal{S}[\varphi_n] dx - \langle F_z, \partial_\nu \mathcal{S}[\varphi_n] \rangle_{-} + (1/2 + \lambda_n) \langle F_z, \varphi_n \rangle \\ &= \omega^2 \int_{\Omega} F_z \mathcal{S}[\varphi_n] dx + \langle F_z, \varphi_n \rangle. \end{aligned}$$

Since $F_z(x) = a \cdot \nabla_z \Gamma^\omega(x - z)$, we have

$$\langle F_z, \varphi_n \rangle = a \cdot \nabla \mathcal{S}^\omega[\varphi_n](z).$$

By (3.6) we have

$$\nabla \mathcal{S}^\omega[\varphi_n](z) = \nabla \mathcal{S}[\varphi_n](z) + O(\omega^2),$$

and hence

$$a_n = a \cdot \nabla \mathcal{S}[\varphi_n](z) + O(\omega^2)$$

Note that $a \cdot \nabla \mathcal{S}[\varphi_n](z)$ is a harmonic function in $z \in \mathbb{R}^3 \setminus \overline{\Omega}$. So it cannot be zero for z in an open set. We choose z so that $a \cdot \nabla \mathcal{S}[\varphi_n](z) \neq 0$, and then $a_n \neq 0$ if ω is sufficiently small. Thus we have

$$\|\nabla u_\delta\|_{L^2(\Omega)} \gtrsim \delta^{-1}. \quad (4.21)$$

This completes the proof. \square

4.2 Two dimensions

In two dimensions we decompose A_δ^s in (4.6) into three parts: $A_\delta^s = A_\delta + B^s + T_\delta^s$ where A_δ is defined by (4.8) and

$$B^s = \begin{bmatrix} \tau^{k_c} \langle \cdot, 1 \rangle & -\tau \langle \cdot, 1 \rangle \\ 0 & 0 \end{bmatrix}. \quad (4.22)$$

Here, τ^{k_c} is defined by

$$\tau^{k_c} = (1/2\pi) (\ln k_c + \gamma - \ln 2) - i/4, \quad (4.23)$$

and τ is defined by (3.5). We emphasize that

$$|\tau^{k_c}| \sim -\ln \omega, \quad |\tau| \sim -\ln \omega. \quad (4.24)$$

We have from (3.13), (3.16) and Lemma 3.1

$$\|T_\delta^s\|_{\mathcal{L}(X,Y)} \lesssim |s^2 \ln s|. \quad (4.25)$$

Unlike the three dimensional case, $A_\delta : X \rightarrow Y$ may not be invertible since $\mathcal{S} : \mathcal{H}^* \rightarrow \mathcal{H}$ is not invertible in general. Instead we prove that $A_\delta + B^s : X \rightarrow Y$ is invertible. In fact, we obtain the following lemma.

Lemma 4.4. *The operator $A_\delta + B^s : X \rightarrow Y$ is invertible. For $(f, g)^T \in Y$, the solution $(\varphi, \psi)^T$ to the equation*

$$(A_\delta + B^s) \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

is given by

$$\varphi = \varphi' + \hat{\varphi}(0)\varphi_0 = \varphi' + \frac{\check{f}(0)\tilde{\mathcal{S}}[\varphi_0] - \hat{g}(0)(\mathcal{S}[\varphi_0] + \tau \langle \varphi_0, 1 \rangle)}{\mathcal{S}[\varphi_0] + \tau^{k_c} \langle \varphi_0, 1 \rangle} \varphi_0 \quad (4.26)$$

and

$$\psi = \varphi' - \mathcal{S}^{-1}[f'] - \hat{g}(0)\varphi_0, \quad (4.27)$$

where

$$\varphi' = \sum_{n=1}^{\infty} \frac{\hat{g}(n) - (\frac{1}{2} + \lambda_n)\check{f}(n)}{(\epsilon_c - 1)(\lambda_n - \lambda(\epsilon_c)) + i\delta(\lambda_n - \frac{1}{2})} \varphi_n. \quad (4.28)$$

Before proving Theorem 4.4, we emphasize that $\mathcal{S}[\varphi_0]$ is constant ($= c_0$). If $c_0 \neq 0$, then $\tilde{\mathcal{S}}[\varphi_0] = \mathcal{S}[\varphi_0] = c_0$, and

$$\langle \varphi_0, 1 \rangle = c_0^{-1} \langle \varphi_0, \mathcal{S}[\varphi_0] \rangle = c_0^{-1}.$$

So we have

$$\hat{\varphi}(0) = \frac{c_0 \check{f}(0) - (c_0 + c_0^{-1} \tau) \hat{g}(0)}{c_0 + c_0^{-1} \tau^{k_c}}.$$

If $c_0 = 0$, then $\tilde{\mathcal{S}}[\varphi_0] = 1$, and $\langle \varphi_0, 1 \rangle = 1$. So we have

$$\hat{\varphi}(0) = \frac{\check{f}(0) - \tau \hat{g}(0)}{\tau^{k_c}}.$$

Proof of Lemma 4.4. Let

$$f = f' + \check{f}(0)\tilde{\mathcal{S}}[\varphi_0] \quad g = g' + \hat{g}(0)\varphi_0$$

be orthogonal decompositions in \mathcal{H} and \mathcal{H}^* , so that $f' \in \mathcal{H}_0$ and $g' \in \mathcal{H}_0^*$.

Since $\mathcal{S} : \mathcal{H}_0^* \rightarrow \mathcal{H}_0$ is invertible, one can see as in Lemma 4.1 that the solution to

$$A_\delta \begin{bmatrix} \varphi' \\ \psi' \end{bmatrix} = \begin{bmatrix} f' \\ g' \end{bmatrix}$$

is given by (4.28) and

$$\psi' = \varphi' - \mathcal{S}^{-1}[f'].$$

Since $(-1/2I + \mathcal{K}^*)[\varphi_0] = 0$ and $\varphi', \psi' \in \mathcal{H}_0^*$, we can see that

$$\begin{aligned} (A_\delta + B^s) \begin{bmatrix} \varphi' + c\varphi_0 \\ \varphi' - \mathcal{S}^{-1}[f'] + d\varphi_0 \end{bmatrix} &= \begin{bmatrix} f' \\ g' \end{bmatrix} + (A_\delta + B^s) \begin{bmatrix} c\varphi_0 \\ d\varphi_0 \end{bmatrix} \\ &= \begin{bmatrix} f' \\ g' \end{bmatrix} + \begin{bmatrix} c(\mathcal{S}[\varphi_0] + \tau^{k_c}\langle \varphi_0, 1 \rangle) - d(\mathcal{S}[\varphi_0] + \tau\langle \varphi_0, 1 \rangle) \\ -d\varphi_0 \end{bmatrix}. \end{aligned}$$

So we solve

$$c(\mathcal{S}[\varphi_0] + \tau^{k_c}\langle \varphi_0, 1 \rangle) - d(\mathcal{S}[\varphi_0] + \tau\langle \varphi_0, 1 \rangle) = \check{f}(0)\tilde{\mathcal{S}}[\varphi_0], \quad -d = \hat{g}(0)$$

for c, d to have (4.26) and (4.27). This completes the proof. \square

We can obtain from Lemma 4.4 results similar to those in Corollary 4.2 for two dimensions. We then obtain the following theorem for two dimensions.

Theorem 4.5. *Suppose $d = 2$ and assume*

$$s^2 |\ln s| \delta^{-1} \leq c \quad (4.29)$$

for sufficiently small c . Let u_δ be the solution to (1.5).

(i) If $\lambda(\epsilon_c/\epsilon_m) \neq \lambda_n$ for any n , then there is C independent of δ (may depend on ϵ_c/ϵ_m) such that

$$\|\nabla u_\delta\|_{L^2(\Omega)} \leq C. \quad (4.30)$$

(iii) If $\lambda(\epsilon_c/\epsilon_m) = \lambda_n$ for some $n \neq 0$, let z be such that $a \cdot \nabla \mathcal{S}[\varphi_n](z) \neq 0$. Then

$$\|\nabla u_\delta\|_{L^2(\Omega)} \approx \delta^{-1} \quad \text{as } \delta \rightarrow +0. \quad (4.31)$$

Proof. We write

$$A_\delta^s = A_\delta + B^s + T_\delta^s = (A_\delta + B^s) (I + (A_\delta + B^s)^{-1} T_\delta^s), \quad (4.32)$$

and follow the same lines of the proof for Theorem 4.3. One thing we need to check is that $|\hat{\varphi}_\delta(0)|$ is bounded. To do that it suffices to show that $|\hat{\varphi}(0)|$ is bounded where φ is the solution expressed in (4.26). Note that

$$|\hat{\varphi}(0)| = \left| \frac{\check{f}(0)\tilde{\mathcal{S}}[\varphi_0] - \hat{g}(0)(\mathcal{S}[\varphi_0] + \tau\langle \varphi_0, 1 \rangle)}{\mathcal{S}[\varphi_0] + \tau^{k_c}\langle \varphi_0, 1 \rangle} \right| \lesssim \frac{|\tau^{k_c}|}{|\tau|} \lesssim 1.$$

This completes the proof. \square

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